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Electro-magneto-phoresis of slender bodies

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Slender-body asymptotic theory is applied to determine the electro-magneto-phoretic motion of a freely suspended elongated particle which is arbitrarily oriented relative to uniformly applied electric and magnetic fields.

1. Introduction

It is well known that the rigid-body motion of a freely suspended particle embedded in a conducting Newtonian fluid can be remotely controlled by externally applied electric and magnetic fields through an electro-magneto-phoretic mechanism. This ability to manipulate the six-velocity motion of a body provides an efficient method for various engineering applications such as impurity extraction, species separation, mixing and stirring, and particle manipulation and control. A review of electro-magneto-phoretic bio-engineering applications was recently given by Watarai, Suwa & Iiguni (2004).

The electro-magneto-phoretic mechanism, which is due to a rotational Lorentz body force, was firstly introduced by Kolin (1953). An analysis for spherical particles was carried out by Leenov & Kolin (1954), and a comparable analysis for ellipsoidal particles was performed by Sellier (2003*a*). A general analysis for arbitrary body shapes was presented by Moffatt & Sellier (2002) using symmetry arguments. Exploiting the bilinear dependence of the Lorentz force-density term upon the electric and magnetic fields, together with the linearity of the Stokes equations, Moffatt & Sellier derived general mobility-type relations for isotropic, axisymmetric and orthotropic particle shapes.

While the analysis of Moffatt & Sellier (2002) provides the tensorial structure for the hydrodynamic forces which act upon non-isotropic particles, it does not give the respective numerical coefficients. In a follow-up paper (Sellier 2003*b*), a general boundary-integral formulation scheme is presented. This scheme enables the calculation of these forces without the need to directly solve the electrostatic and flow problems. This formulation, which only requires prescribing the value of the potential and its derivatives on the particle surface, is natural for use in numerical analyses. It also renders analytic expressions for ellipsoidal particles.

The symmetry analysis of Moffatt & Sellier (2002) demonstrated that the combination of electric and magnetic fields can result in a rich topology of particle motion, unparalleled by other (e.g. phoretic) animation mechanisms. Since highly symmetric particle shapes do not exhibit that richness, it is desirable to analyse more general shapes, even in an approximate manner: such approximations can be used for understanding the dynamics and control of non-isotropic particles. Unfortunately, the comparable calculation for non-spherical shapes presents a formidable task. In this work we consider slender particle shapes, which enable the derivation of asymptotic approximations. Since slender particles commonly appear in colloidal and biological systems, the present analysis could serve (together with the boundary-integral formulation of Sellier 2003*b*) as a modelling tool in various branches of applied research. Indeed, slender bodies have already been analysed in other field-driven applications, such as fixed-charge (Solomentsev & Anderson 1994; Sellier 2000) and induced-charge (Saintillan, Darve & Shaqfeh 2006) electrophoresis.

In the present work, we employ the common assumption (Sellier 2003*b*) of particle magnetic permeability which is identical to that of the ambient liquid. Thus, while the particle modifies the electric field in the fluid due to its different conductivity, it does not affect the magnetic field. For small particles, on the scale of 1 mm or less, the Hartmann number is small (Moffatt & Sellier 2002), whence the current is proportional to the electric field. In that limit, the electrostatic problem is governed by the standard exterior Neumann problem, which is decoupled from the hydrodynamic problem.

Since the solution for the harmonic Neumann problem about slender bodies is well-known (see e.g. Thwaites 1960), the difficulty amounts to dealing with the hydrodynamics, which for small-particle applications is governed by the creeping-flow equations. Following Leenov & Kolin (1954), the flow problem is decomposed into three parts. The first takes account of the Lorentz rotational body force, the second is introduced so as to retain mass conservation, and the third is used to maintain the impermeability and no-slip boundary conditions on the particle surface. Using an axial dipole-distribution representation for the electric potential, we obtain analytic solutions for the first and second parts. The solution to the third part is then readily obtained by making use of the Lorentz reciprocal theorem (Happel & Brenner 1965).

Using this approach, we present a general programme for the evaluation of the force and torque acting on a stationary axisymmetric particle for arbitrary orientations of the applied fields relative to the particle axis. For a freely suspended particle, the rectilinear and rotational velocities may be obtained using the appropriate mobility relations (Kim & Karrila 1991). Motivated by the symmetry arguments of Moffatt & Sellier (2002), the subsequent analysis is demonstrated for two configurations. In the first, the electric field, magnetic field, and particle axis are mutually orthogonal. This results in an axial particle migration. In the second, all three directions are parallel; for body shapes that lack fore–aft symmetry, this results in a rotation of the body about its axis.

2. Problem formulation

An insulating particle of characteristic dimension a is positioned within an unbounded Newtonian liquid of viscosity η, electrical conductivity σ, and matching magnetic permeability. The fluid domain is denoted by $\mathscr{D}$, and the particle surface by $\mathscr{S}$. This system is exposed to uniformly applied electric field $\boldsymbol{E} = E\hat{\boldsymbol{E}}$ and magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{B}}$, where $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ denote the respective unit vectors. Our interest lies in the resulting hydrodynamic force and torque exerted on the particle, and, consequently, in the animated motion a freely suspended particle would experience.

2.1. *Governing equations*

We use dimensionless notation, wherein length variables (and the gradient operator) are scaled using a, the electric field with E, and electrical potential with aE. The

Lorentz body force implies the stress scale σEBa (whence the respective scales σEBa^3 and σEBa^4 for forces and torques), as well as the velocity scale $\sigma EBa^2/\eta$. Angular velocities are accordingly normalized with $\sigma EBa/\eta$.

The presence of the particle modifies the electric field from its undisturbed value $\hat{\boldsymbol{E}}$ to $\hat{\boldsymbol{E}} - \nabla\varphi$, where the potential disturbance φ is governed by the the following exterior Neumann problem:

$$\nabla^2\varphi = 0 \quad \text{in} \quad \mathscr{D}, \qquad \hat{\boldsymbol{n}} \cdot \nabla\varphi = \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{E}} \quad \text{on} \quad \mathscr{S}, \qquad \varphi \to 0 \quad \text{as} \quad |\boldsymbol{x}| \to \infty \tag{2.1}$$

(wherein $\hat{\boldsymbol{n}}$ is an outward-pointing unit vector normal to $\mathscr{S}$). This field results in the Lorentz force density distribution

$$\hat{\boldsymbol{E}} \times \hat{\boldsymbol{B}} - \nabla\varphi \times \hat{\boldsymbol{B}}. \tag{2.2}$$

The first term in (2.2) represents a uniform body force; it is balanced by a hydrostatic-type pressure distribution, which in turn results in the electromagnetic buoyancy force

$$-\mathscr{V}\, \hat{\boldsymbol{E}} \times \hat{\boldsymbol{B}}, \tag{2.3}$$

and torque

$$-\mathscr{V}\, \boldsymbol{x}_C \times (\hat{\boldsymbol{E}} \times \hat{\boldsymbol{B}}) \tag{2.4}$$

which act upon the particle. Here, $\mathscr{V}$ is the particle volume (normalized with a^3) and $\boldsymbol{x}_C$ is its centroid position vector.

In general, the second term in (2.2) is rotational, and therefore cannot be balanced by any pressure distribution; thus, it results in a fluid motion. This motion is governed by the continuity

$$\nabla \cdot \boldsymbol{v} = 0 \tag{2.5}$$

and the inhomogeneous Stokes

$$\nabla^2\boldsymbol{v} = \nabla p + \nabla\varphi \times \hat{\boldsymbol{B}} \tag{2.6}$$

equations. Here, $\boldsymbol{v}$ is the velocity field and p is the 'modified' pressure, additional to the 'hydrostatic' distribution which generates (2.3) and (2.4). These equations are supplemented by the impermeability and no-slip boundary conditions, which for a stationary particle are

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \mathscr{S}, \tag{2.7}$$

together with the requirement that $\boldsymbol{v}$ decays to zero at large distances from the particle. We seek to obtain analytic expressions for the hydrodynamic force and torque engendered by this flow. The case of a freely suspended particle is then readily obtained by making use of appropriate mobility relations.

2.2. *Decomposition scheme*

Following Leenov & Kolin (1954), the velocity field is decomposed into three sub-fields. The first is a particular integral of the inhomogeneous Stokes equation (2.6):

$$\nabla^2\boldsymbol{v}_1 = \nabla\varphi \times \hat{\boldsymbol{B}}. \tag{2.8}$$

In general, $\boldsymbol{v}_1$ is not solenoidal; to satisfy the continuity equation (2.5), it is corrected by a second sub-field which satisfies

$$\nabla \cdot \boldsymbol{v}_2 = -\nabla \cdot \boldsymbol{v}_1, \quad \nabla^2\boldsymbol{v}_2 = \boldsymbol{0}. \tag{2.9}$$

To satisfy the boundary condition on $\mathscr{S}$, the introduction of a third sub-field is required. It satisfies the standard Stokes equations,

$$\nabla \cdot \boldsymbol{v}_3 = 0, \quad \nabla^2 \boldsymbol{v}_3 = \nabla p. \tag{2.10}$$

together with the 'slip' condition

$$\boldsymbol{v}_3 = -\boldsymbol{v}_1 - \boldsymbol{v}_2 \quad \text{on} \quad \mathscr{S}. \tag{2.11}$$

All three sub-fields must attenuate to zero at large distances from the particle.

Once $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are calculated, there is no need to solve for $(\boldsymbol{v}_3, p)$, since the force and torque it delivers can be calculated indirectly using the reciprocal theorem (Happel & Brenner 1965). According to that theorem, any two flow fields, say $\boldsymbol{v}'$ and $\boldsymbol{v}''$, that satisfy the Stokes equations (2.10) in a domain $\mathscr{D}$ with boundary $\partial\mathscr{D}$, are related by

$$\oint_{\partial\mathscr{D}} \mathrm{d}A\, \hat{\boldsymbol{n}} \cdot \boldsymbol{S}' \cdot \boldsymbol{v}'' = \oint_{\partial\mathscr{D}} \mathrm{d}A\, \hat{\boldsymbol{n}} \cdot \boldsymbol{S}'' \cdot \boldsymbol{v}'. \tag{2.12}$$

Here, $\boldsymbol{S}'$ and $\boldsymbol{S}''$ denote the respective stress fields which correspond to $\boldsymbol{v}'$ and $\boldsymbol{v}''$, and $\hat{\boldsymbol{n}}$ points into $\mathscr{D}$. In the present configuration, where $\mathscr{D}$ is the domain outside the particle, the contribution to the integrals arises from the particle surface, $\partial\mathscr{D} = \mathscr{S}$. Following the method of Brenner (1964), we choose $\boldsymbol{v}'$ as $\boldsymbol{v}_3$ and $\boldsymbol{v}''$ as a field which corresponds to a pure translation or rotation of the particle. This procedure yields the desired expressions (expressed as surface quadratures) for the hydrodynamic loads engendered by $\boldsymbol{v}_3$.

Using this decomposition approach, Leenov & Kolin (1954) calculated the force acting on a spherical particle. (Having published their work a decade before the techniques of Brenner (1964) became available, Kolin & Leenov actually went to the trouble of calculating $\boldsymbol{v}_3$ directly.) Here we consider slender bodies, for which simple analytic approximations can be obtained using a rational asymptotic scheme.

3. Slender bodies

We consider axisymmetric bodies, described by a unit vector $\hat{\boldsymbol{e}}$ attached to the symmetry axis. This vector may be arbitrarily oriented relative to $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$. We employ a body-fixed Cartesian coordinate system xyz with z-axis pointing in the $\hat{\boldsymbol{e}}$-direction and x-axis in the direction of the transverse component of $\hat{\boldsymbol{E}}$. The angle between $\hat{\boldsymbol{e}}$ and $\hat{\boldsymbol{E}}$ is denoted by θ. It is also convenient to employ the cylindrical coordinate $r = (x^2 + y^2)^{1/2}$ and the azimuthal angle ϖ about the z-axis. With no loss of generality, the body extends between $z = -1$ and $z = 1$. Its boundary is given by $r = \epsilon R(z)$ where $R(z)$ is an $O(1)$ shape function which satisfies

$$R(\pm 1) = 0. \tag{3.1}$$

A schematic of such a body is presented in figure 1.

We focus upon slender shapes for which $\epsilon \ll 1$. It is well known that some of the expansions in slender-body theory involve powers of the small parameter $1/\ln\epsilon$; in that case, a single term in such expansions is only of semi-qualitative value (Tuck 1964), and a true leading-order asymptotic approximation requires two successive terms in such powers. As is quite standard in slender-body analyses (Cox 1970), we neglect algebraically small terms in ϵ.

For slender bodies, the electrostatic problem (2.1) has been solved (Thwaites 1960; Hinch 1991) by representing φ as a dipole distribution along the particle centreline.

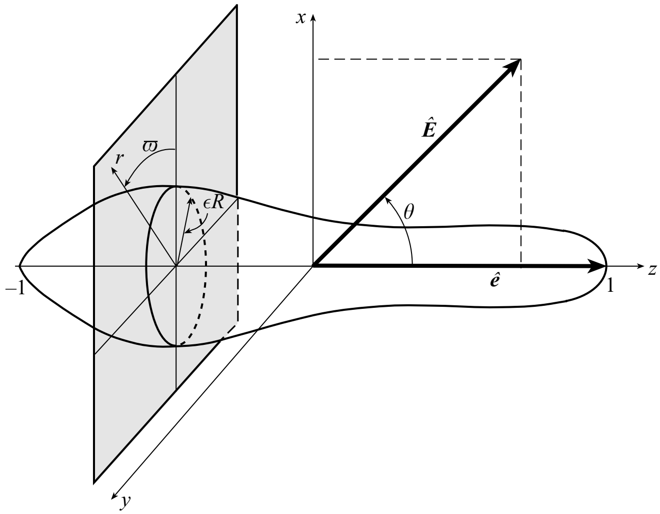

FIGURE 1. A schematic of a 'pear-shaped' axisymmetric particle. The vector $\hat{\boldsymbol{B}}$ can have any orientation relative to the axes xyz.

Neglecting algebraically small terms:

$$\varphi = -\frac{\epsilon^2}{4}\cos\theta\int_{-1}^{1} R^2(z_0)\,\frac{z-z_0}{s^3}\,\mathrm{d}z_0 - \frac{\epsilon^2}{2}\sin\theta\int_{-1}^{1} R^2(z_0)\,\frac{x}{s^3}\,\mathrm{d}z_0. \tag{3.2}$$

The two integrals represent the respective contributions to φ from the longitudinal and transverse components of $\hat{\boldsymbol{E}}$. Noting that $\hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}$, we combine these two integrals into the invariant form

$$\varphi = -\epsilon^2\left(\int_{-1}^{1} R^2(z_0)\,\frac{\boldsymbol{s}}{2s^3}\,\mathrm{d}z_0\right)\cdot\left(\boldsymbol{I} - \tfrac{1}{2}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\right)\cdot\hat{\boldsymbol{E}}. \tag{3.3}$$

Here, $\boldsymbol{I}$ denotes the idemfactor, $\boldsymbol{s} = (z-z_0)\,\hat{\boldsymbol{e}}_z + r\hat{\boldsymbol{e}}_r$ is a relative position vector, and $s = |\boldsymbol{s}|$.

The mathematical expression for φ about a slender body is similar to that appropriate for a spherical particle: the former is represented by a dipole distribution, and the latter by a single dipole (Leenov & Kolin 1954). This resemblance leads us to obtain a closed-form solution for (2.8),

$$\boldsymbol{v}_1 = \frac{\epsilon^2}{4}\hat{\boldsymbol{B}}\times\left(\int_{-1}^{1} R^2(z_0)\,\frac{\boldsymbol{s}\boldsymbol{s}}{s^3}\,\mathrm{d}z_0\right)\cdot\left(\boldsymbol{I} - \tfrac{1}{2}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\right)\cdot\hat{\boldsymbol{E}}, \tag{3.4}$$

as well as one for (2.9),

$$\boldsymbol{v}_2 = -\frac{\epsilon^2}{4}\hat{\boldsymbol{B}}\times\left(\boldsymbol{I} - \tfrac{1}{2}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}\right)\cdot\hat{\boldsymbol{E}}\int_{-1}^{1}\frac{R^2(z_0)}{s}\,\mathrm{d}z_0. \tag{3.5}$$

These two key expressions, which can be verified by substitution into (2.8) and (2.9), allow the analytical treatment of slender bodies.

The calculation of $\boldsymbol{v}_1$ requires evaluating the following integrals

$$\left.\begin{aligned} I_{1,1}(r,z;\epsilon) &= \int_{-1}^{1} \frac{R^2(z_0)}{s^3}\,\mathrm{d}z_0, \\ I_{1,2}(r,z;\epsilon) &= \int_{-1}^{1} R^2(z_0)\,\frac{z-z_0}{s^3}\,\mathrm{d}z_0, \\ I_{1,3}(r,z;\epsilon) &= \int_{-1}^{1} R^2(z_0)\,\frac{(z-z_0)^2}{s^3}\,\mathrm{d}z_0, \end{aligned}\right\} \tag{3.6}$$

while for the calculation of $\boldsymbol{v}_2$ we need to evaluate the integral

$$I_2(r,z;\epsilon) = \int_{-1}^{1} \frac{R^2(z_0)}{s}\,\mathrm{d}z_0. \tag{3.7}$$

Once I_2 is determined, the integrals (3.6) can be directly calculated by using the relations

$$I_{1,1} = -\frac{1}{r}\frac{\partial I_2}{\partial r}, \quad I_{1,2} = -\frac{\partial I_2}{\partial z}, \quad I_{1,3} = I_2 + r\frac{\partial I_2}{\partial r}. \tag{3.8}$$

We evaluate I_2 near the particle, where $r=\epsilon\rho$ with $\rho=O(1)$. For $\epsilon \to 0$ one can verify that

$$I_2(\rho,z;\epsilon) = 2R^2(z)\ln(2/\epsilon) + R^2(z)\ln\frac{1-z^2}{\rho^2} + f(z) + O(\epsilon^2) \tag{3.9}$$

where

$$f(z) = \int_{-1}^{1} \frac{R^2(z_0)-R^2(z)}{|z_0-z|}\,\mathrm{d}z_0 \tag{3.10}$$

is an $O(1)$ function which depends upon the body shape. Note that $f(z)$ is an even function for bodies which possess fore–aft symmetry.

Thus, the calculation of the net hydrodynamic force and torque exerted on the body requires three steps: (i) evaluating the force and torque delivered by the stresses engendered by the fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (since $\nabla\cdot\boldsymbol{v}_2=-\nabla\cdot\boldsymbol{v}_1$, these stresses may be evaluated based upon the Newtonian expressions for an incompressible fluid); (ii) evaluate $\boldsymbol{v}_3=-(\boldsymbol{v}_1+\boldsymbol{v}_2)$ on $\mathscr{S}$, where $\rho=R(z)$; (iii) Use the reciprocal theorem (2.12), in conjunction with existing solutions for flows driven by rigid-body motion of slender bodies, to evaluate the force and torque delivered by $\boldsymbol{v}_3$.

Systematic asymptotic analyses of Stokes flows due to the rigid-body motion of slender shapes were published at about the same time by Tillett (1970), Batchelor (1970), and Cox (1970). For our purpose, the latter analysis is most suitable: Cox has employed inner–outer expansions, wherein the body appears as an infinite cylinder in the inner analysis and as a Stokeslet line distribution in the outer one. The velocity fields that correspond to rectilinear and rotational particle (which are to be used as $\boldsymbol{v}''$ in (2.12)), as well as the corresponding stress fields (to be used as $\boldsymbol{S}''$), are readily obtained from Cox's inner solution.

In principle, the three-step solution procedure can be applied for any arbitrary triad $(\hat{\boldsymbol{E}}, \hat{\boldsymbol{B}}, \hat{\boldsymbol{e}})$. In what follows, we focus our attention upon two important cases. The first is a conventional configuration where these three vectors are mutually orthogonal. In that configuration, Moffatt & Sellier (2002) showed that an axisymmetric particle cannot experience any torque, but can nevertheless experience a force along the direction $\hat{\boldsymbol{E}} \times \hat{\boldsymbol{B}}$, just like a spherical particle. In the second case, all three vectors are collinear. For that case, Moffatt & Sellier (2002) showed that an axisymmetric particle cannot experience any force, but can experience a couple about its axis, provided it lacks fore–aft symmetry. Since the translation and rotation of an axisymmetric particle are uncoupled in these configurations, a freely suspended particle will migrate along its axis in the first scenario, and rotate about its axis in the second.

4. The case of mutually orthogonal $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{B}}$, and $\hat{\boldsymbol{e}}$

We consider here the case when the vectors $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{B}}$, and $\hat{\boldsymbol{e}}$ are mutually orthogonal, say $\hat{\boldsymbol{E}} = \hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{B}} = \hat{\boldsymbol{e}}_y$. For that geometry, the symmetry arguments of Moffatt & Sellier (2002) imply that the particle may experience a force only in the z-direction, and no torque.

The representations (3.4) and (3.5) imply that $\boldsymbol{v}_1 = \hat{\boldsymbol{e}}_z w_1$ and $\boldsymbol{v}_2 = \hat{\boldsymbol{e}}_z w_2$, where

$$w_1(\rho, z; \epsilon) = \frac{\epsilon^3}{4}(\rho I_{1,2} \cos\varpi - \epsilon\rho^2 I_{1,1} \cos^2\varpi), \quad w_2(\rho, z; \epsilon) = \frac{\epsilon^2}{4} I_2.$$

Use of (3.8)–(3.9) furnishes the combined velocity near the particle surface, up to algebraically small terms:

$$w_1 + w_2 = \frac{\epsilon^2}{4}\left[2R^2(z)\ln(2/\epsilon) + R^2(z)\ln\frac{1-z^2}{\rho^2} + f(z) - 2R^2(z)\cos^2\varpi\right]. \tag{4.1}$$

When evaluating the force delivered by this velocity in the z-direction, $F_1 + F_2$, we note that $\hat{\boldsymbol{n}}$ points (up to algebraically small terms) in the radial direction. Accordingly, only the rz-component of the combined stress due to that velocity is needed for a leading-order force calculation. This shear stress,

$$\frac{1}{\epsilon}\frac{\partial(w_1 + w_2)}{\partial\rho}\bigg|_{\rho=R(z)} \simeq -\frac{\epsilon R(z)}{2},$$

results in the force

$$F_1 + F_2 = -\pi\epsilon^2 \int_{-1}^{1} R^2(z)\,\mathrm{d}z. \tag{4.2}$$

The boundary condition governing $\boldsymbol{v}_3$ adopts the form $\boldsymbol{v}_3|_{\rho=R(z)} = \hat{\boldsymbol{e}}_z w_3(z)$, where

$$\begin{aligned} w_3(z) &= -[w_1 + w_2]_{\rho=R(z)} \\ &= -\frac{R^2(z)}{2}\epsilon^2\ln(1/\epsilon) - \epsilon^2\frac{R^2(z)}{2}\left[\ln 2 + \frac{1}{2}\ln\frac{1-z^2}{R^2(z)} - \cos^2\varpi + \frac{f(z)}{2R^2(z)}\right]. \end{aligned} \tag{4.3}$$

The force delivered by $\boldsymbol{v}_3$ in the z-direction, F_3, is next calculated using the reciprocal theorem (2.12). Here we choose $\boldsymbol{v}''$ as a field which corresponds to particle translation along the z-axis with a unit velocity. Ignoring algebraically small terms, we then find

$$F_3 = \oint_{\mathscr{S}} \mathrm{d}A\, S''_{rz}(z) w_3(z), \tag{4.4}$$

where S''_{rz} is the rz-component of $\boldsymbol{S}''$. Owing to the axisymmetric nature of the translation problem, this shear stress is given by $\mathscr{F}/2\pi\epsilon R(z)$, where $\mathscr{F}$ is the respective force per unit length acting on the particle. The force distribution $\mathscr{F}(z)$ was evaluated by Cox (1970) and in the present notation is given as

$$\mathscr{F}(z) = -2\pi\left[\frac{1}{\ln(1/\epsilon)} + \frac{1}{\ln^2(1/\epsilon)}\left(\frac{1}{2} - \ln 2 - \frac{1}{2}\ln\frac{1-z^2}{R^2(z)}\right) + O(\ln^{-3}\epsilon)\right]. \tag{4.5}$$

Substitution of (4.3) and (4.5) into (4.4) yields

$$F_3 = \pi\epsilon^2 \left[\int_{-1}^{1} R^2(z)\,\mathrm{d}z + \frac{1}{2\ln(1/\epsilon)} \int_{-1}^{1} f(z)\,\mathrm{d}z + O(\ln^{-2}\epsilon) \right]. \tag{4.6}$$

In addition to the force delivered by the three sub-fields, the particle also experiences an electromagnetic buoyancy force (2.3) which here acts in the z-direction with a magnitude

$$-\pi\epsilon^2 \int_{-1}^{1} R^2(z)\,\mathrm{d}z. \tag{4.7}$$

Thus, the total hydrodynamic force, obtained by summing (4.2), (4.6), and (4.7), is given by

$$-\pi\epsilon^2 \left[\int_{-1}^{1} R^2(z)\,\mathrm{d}z - \frac{1}{2\ln(1/\epsilon)} \int_{-1}^{1} f(z)\,\mathrm{d}z + O(\ln^{-2}\epsilon) \right]. \tag{4.8}$$

If the particle is freely suspended, it will migrate in the z-direction with a velocity given by the ratio of (4.8) to the resistance coefficient for longitudinal translation. This coefficient is readily given by (see (4.5))

$$4\pi \left\{ \frac{1}{\ln(1/\epsilon)} + \frac{1}{\ln^2(1/\epsilon)} \left[\frac{1}{2} - \ln 2 - \frac{1}{4} \int_{-1}^{1} \ln \frac{1-z^2}{R^2(z)}\,\mathrm{d}z \right] + O(\ln^{-3}\epsilon) \right\}. \tag{4.9}$$

Thus, we have obtained an asymptotic approximation for the migration velocity, which may be evaluated for any prescribed particle shape. For example, for the case of a spheroid, $R(z) = (1 - z^2)^{1/2}$, we obtain the migration speed

$$-\tfrac{1}{3}\epsilon^2 \left[\ln(1/\epsilon) - \left(\tfrac{1}{2} - \ln 2 \right) + O(\ln^{-1}\epsilon) \right] \tag{4.10}$$

which agrees with results (5.17)–(5.18) of Sellier (2003*b*).†

5. The case of collinear $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{B}}$, and $\hat{\boldsymbol{e}}$

Next, we focus on the case where the three vectors $\hat{\boldsymbol{E}}$, $\hat{\boldsymbol{B}}$, and $\hat{\boldsymbol{e}}$ are collinear. Since the 'hydrostatic' force density $\hat{\boldsymbol{E}} \times \hat{\boldsymbol{B}}$ vanishes, one could naively think that this configuration may not lead to any particle motion (which is indeed the case for a spherical particle). However, Moffatt & Sellier (2002) indicated that axisymmetric particles which lack fore–aft symmetry may experience a net couple about their axis. This possibility is explored herein.

The representation (3.4) implies that $\boldsymbol{v}_1 = \hat{\boldsymbol{e}}_{\varpi} v_1$, where $v_1 = \epsilon^3 \rho I_{1,2}/8$. Since any axisymmetric azimuthal flow automatically satisfies the continuity equation (2.5), no need arises for introducing the complementary sub-field $\boldsymbol{v}_2$ (note, indeed, the vanishing of (3.5)). Use of (3.8)–(3.9) furnishes the following approximation for v_1 near the particle surface:

$$v_1 = \frac{\epsilon^3}{8}\rho \left[-4R\frac{\mathrm{d}R}{\mathrm{d}z}\ln(2/\epsilon) - 2R\frac{\mathrm{d}R}{\mathrm{d}z}\ln\frac{1-z^2}{\rho^2} + \frac{2zR^2}{1-z^2} - \frac{\mathrm{d}f}{\mathrm{d}z} + O(\epsilon^2) \right]. \tag{5.1}$$

Only the $z\varpi$-component of the stress tensor generated by $\boldsymbol{v}_1$,

$$\epsilon^{-1}\rho \frac{\partial}{\partial\rho}\left(\frac{v_1}{\rho}\right)\bigg|_{\rho=R(z)} \simeq \frac{1}{2}\epsilon^2 R\frac{\mathrm{d}R}{\mathrm{d}z},$$

† In equation (5.18) of Sellier (2003*b*) the $\ln 2$ term was overlooked, as can be verified from scrutiny of equations (5.4)–(5.6) in that paper.

is relevant for the evaluation of the leading-order axial torque. This shear stress results in a torque per unit length of magnitude $\pi\epsilon^4 R^3\,\mathrm{d}R/\mathrm{d}z$. The end-conditions (3.1), however, imply that it integrates out to produce a null $O(\epsilon^4)$ torque. Thus, the torque delivered by $\boldsymbol{v}_1$ is $o(\epsilon^4)$. As will be readily seen, this torque is dominated by that delivered by $\boldsymbol{v}_3$.

Since $\boldsymbol{v}_1$ is azimuthal, so must be $\boldsymbol{v}_3$: thus $\boldsymbol{v}_3 = \hat{\boldsymbol{e}}_\varpi v_3$. On the particle surface, where $\rho = R(z)$, the boundary condition governing $\boldsymbol{v}_3$ adopts the form

$$v_3 = \frac{R^2}{2}\frac{\mathrm{d}R}{\mathrm{d}z}\epsilon^3 \ln(2/\epsilon) + \epsilon^3\left[\frac{R^2}{4}\frac{\mathrm{d}R}{\mathrm{d}z}\ln\frac{1-z^2}{R^2} - \frac{zR^3}{4(1-z^2)} + \frac{R}{8}\frac{\mathrm{d}f}{\mathrm{d}z}\right] + O(\epsilon^5). \tag{5.2}$$

The torque delivered by this field about the z-direction, T_3, is next calculated by referring again to the reciprocal theorem (2.12). We choose $\boldsymbol{v}''$ as a field which corresponds to pure rotation about the z-axis with a unit velocity.

In Cox's (1970) derivation, where algebraic errors are neglected, the velocity along the actual particle surface is replaced with that along its centreline. Thus, axial rotation constitutes a special case for which Cox's method is inapplicable. Luckily, the velocity field that corresponds to axial rotation can be obtained without resorting to more accurate asymptotic approximations (such as Geer 1976): in the vicinity of any point $(\epsilon R(z), \varpi, z)$ on $\mathscr{S}$, the velocity $\boldsymbol{v}''$ is approximately equal to that corresponding to the rotation of an infinite cylinder of (constant) radius $\epsilon R(z)$.† This is a potential vortex of azimuthal velocity profile $\epsilon^2 R^2(z)/r$; the corresponding shear stress on $\mathscr{S}$ is -2. Neglecting algebraically small errors, we then find from (2.12):

$$T_3 = -4\pi\epsilon \int_{-1}^{1} R(z) v_3(z)\,\mathrm{d}z. \tag{5.3}$$

Substitution of (5.2) in conjunction with (3.1) furnishes the expression

$$T_3 = -\pi\epsilon^4 \int_{-1}^{1}\left[R^3\frac{\mathrm{d}R}{\mathrm{d}z}\ln(1-z^2) - \frac{zR^4}{1-z^2} + \frac{R^2}{2}\frac{\mathrm{d}f}{\mathrm{d}z}\right]\mathrm{d}z. \tag{5.4}$$

This $O(\epsilon^4)$ contribution dominates the $o(\epsilon^4)$ contribution delivered by $\boldsymbol{v}_1$. The integral in (5.4) is finite for all continuous shapes that satisfy (3.1); these shapes can be either cusped or rounded at the ends.

For a freely suspended particle, this torque results in an angular velocity. Using the potential-vortex solution for pure rotation, the resistance coefficient for axial rotation is readily obtained as

$$4\pi\epsilon^2 \int_{-1}^{1} R(z)\,\mathrm{d}z.$$

Accordingly, the resulting angular velocity is given by the expression

$$-\epsilon^2\left[4\int_{-1}^{1} R(z)\,\mathrm{d}z\right]^{-1}\int_{-1}^{1}\left[R^3\frac{\mathrm{d}R}{\mathrm{d}z}\ln(1-z^2) - \frac{zR^4}{1-z^2} + \frac{R^2}{2}\frac{\mathrm{d}f}{\mathrm{d}z}\right]\mathrm{d}z. \tag{5.5}$$

Note that this expression vanishes for flip-symmetric particles, in accordance with the prediction of Moffatt & Sellier (2002).

† This result reflects the non-singular nature of the flow due to rigid axial rotation. In the comparable problem of rigid translation, it is impossible to find a creeping-flow solution that satisfies the boundary condition on an infinite cylinder and simultaneously decays at large distances; thus, the inner field near the particle must be obtained via appropriate matching with a comparable outer field (Cox 1970).

The symmetry arguments of Moffatt & Sellier (2002) imply that pure axial rotation would also occur if the body axis were taken to be perpendicular to the applied fields ($\theta = \pi/2$). In the general case, when the axis is oriented at an arbitrary angle θ relative to the applied fields, the axial torque may be accompanied by an additional torque about the direction of the applied fields, causing particle precession. Since neither of these torques affects the value of θ, both the aligned ($\theta = 0$) and the perpendicular ($\theta = \pi/2$) configurations are neutrally stable.

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